

The Construction of Quantum Field Operators: Something of Interest*

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Abstract

We draw attention to some tune problems in constructions of the quantum-field operators for spins $1/2$ and 1 . They are related to the existence of negative-energy and acausal solutions of relativistic wave equations. Particular attention is paid to the chiral theories, and to the method of the Lorentz boosts.

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1 The Dirac Equation.

First of all, I would like to remind you some basic things in the quantum field theory.

The Dirac equation has been considered in detail in a pedagogical way [Sakurai, Ryder]:

$$[i\gamma^\mu\partial_\mu - m]\Psi(x) = 0. \quad (1)$$

At least, 3 methods of its derivation exist:

- the Dirac one (the Hamiltonian should be linear in $\partial/\partial x^\mu$, and be compatible with $E^2 - \mathbf{p}^2 c^2 = m^2 c^4$);
- the Sakurai one (based on the equation $(E - \sigma \cdot \mathbf{p})(E + \sigma \cdot \mathbf{p})\phi = m^2 \phi$);
- the Ryder one (the relation between 2-spinors at rest is $\phi_R(\mathbf{0}) = \pm \phi_L(\mathbf{0})$).

The γ^μ are the Clifford algebra matrices

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}. \quad (2)$$

Usually, everybody uses the following definition of the field operator [Itsykson]:

$$\Psi(x) = \frac{1}{(2\pi)^3} \sum_\sigma \int \frac{d^3\mathbf{p}}{2E_p} [u_\sigma(\mathbf{p})a_\sigma(\mathbf{p})e^{-ip \cdot x} + v_\sigma(\mathbf{p})b_\sigma^\dagger(\mathbf{p})e^{+ip \cdot x}], \quad (3)$$

as given *ab initio*.

I studied in the previous works [Dvoeglazov1, Dvoeglazov2, Dvoeglazov3]:

- $\sigma \rightarrow h$ (the helicity basis);

- the modified Sakurai derivation (the additional $m_2\gamma^5$ term in the Dirac equation);
- the derivation of the Barut equation [Barut] from the first principles, namely based on the generalized Ryder relation, $(\phi_L^h(\mathbf{0}) = \hat{A}\phi_L^{-h*}(\mathbf{0}) + \hat{B}\phi_L^{h*}(\mathbf{0}))$. In fact, we have the second mass state (μ -meson) from that equation:

$$[i\gamma^\mu\partial_\mu - \alpha\partial_\mu\partial^\mu/m - \beta]\psi = 0; \quad (4)$$

- the self/anti-self charge-conjugate Majorana 4-spinors [Majorana, Bilenky] in the momentum representation.

The Wigner rules [Wigner] of the Lorentz transformations for the $(0, S)$ left- $\phi_L(\mathbf{p})$ and the $(S, 0)$ right- $\phi_R(\mathbf{p})$ spinors are:

$$(S, 0) : \quad \phi_R(\mathbf{p}) = \Lambda_R(\mathbf{p} \leftarrow \mathbf{0}) \phi_R(\mathbf{0}) = \exp(+\mathbf{S} \cdot \varphi) \phi_R(\mathbf{0}), \quad (5)$$

$$(0, S) : \quad \phi_L(\mathbf{p}) = \Lambda_L(\mathbf{p} \leftarrow \mathbf{0}) \phi_L(\mathbf{0}) = \exp(-\mathbf{S} \cdot \varphi) \phi_L(\mathbf{0}), \quad (6)$$

with $\varphi = \mathbf{n}\varphi$ being the boost parameters:

$$\cosh(\varphi) = \gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \sinh(\varphi) = \beta\gamma = \frac{v/c}{\sqrt{1 - v^2/c^2}} \quad (7)$$

$$\tanh(\varphi) = v/c. \quad (8)$$

They are well known and given, *e.g.*, in [Wigner, Faustov, Ryder].

On using the Wigner rules and the Ryder relations we can recover the Dirac equation in the matrix form:

$$\begin{pmatrix} \mp m 1 & p_0 + \sigma \cdot \mathbf{p} \\ p_0 - \sigma \cdot \mathbf{p} & \mp m 1 \end{pmatrix} \psi(p^\mu) = 0, \quad (9)$$

or $(\gamma \cdot p - m)u(\mathbf{p}) = 0$ and $(\gamma \cdot p + m)v(\mathbf{p}) = 0$. We have used the property $[\Lambda_{L,R}(\mathbf{p} \leftarrow \mathbf{0})]^{-1} = [\Lambda_{R,L}(\mathbf{p} \leftarrow \mathbf{0})]^\dagger$ above, and that

both \mathbf{S} and $\Lambda_{R,L}$ are Hermitian for the finite $(S = 1/2, 0) \oplus (0, S = 1/2)$ representation of the Lorentz group. Introducing $\psi(x) \equiv \psi(p) \exp(\mp i p \cdot x)$ and letting $p_\mu \rightarrow i\partial_\mu$, the above equation becomes the Dirac equation (1).

The solutions of the Dirac equation are denoted by $u(\mathbf{p}) = \text{column}(\phi_R(\mathbf{p}) \ \phi_L(\mathbf{p}))$ and $v(\mathbf{p}) = \gamma^5 u(\mathbf{p})$. Let me remind that the boosted 4-spinors in the common-used basis (the standard representation of γ matrices) are

$$\begin{aligned} u_{\frac{1}{2}, \frac{1}{2}} &= \sqrt{\frac{(E+m)}{2m}} \begin{pmatrix} 1 \\ 0 \\ p_z/(E+m) \\ p_r/(E+m) \end{pmatrix}, \\ u_{\frac{1}{2}, -\frac{1}{2}} &= \sqrt{\frac{(E+m)}{2m}} \begin{pmatrix} 0 \\ 1 \\ p_l/(E+m) \\ -p_z/(E+m) \end{pmatrix}, \end{aligned} \quad (10)$$

$$\begin{aligned} v_{\frac{1}{2}, \frac{1}{2}} &= \sqrt{\frac{(E+m)}{2m}} \begin{pmatrix} p_z/(E+m) \\ p_r/(E+m) \\ 1 \\ 0 \end{pmatrix}, \\ v_{\frac{1}{2}, -\frac{1}{2}} &= \sqrt{\frac{(E+m)}{2m}} \begin{pmatrix} p_l/(E+m) \\ -p_z/(E+m) \\ 0 \\ 1 \end{pmatrix}. \end{aligned} \quad (11)$$

$E = \sqrt{\mathbf{p}^2 + m^2} > 0$, $p_0 = \pm E$, $p^\pm = E \pm p_z$, $p_{r,l} = p_x \pm ip_y$. They are the parity eigenstates with the eigenvalues of ± 1 . In the parity operator the matrix $\gamma_0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ was used as usual. They also describe eigenstates of the charge operator, Q , if at

rest

$$\phi_R(\mathbf{0}) = \pm \phi_L(\mathbf{0}) \quad (12)$$

(otherwise the corresponding physical states are no longer the charge eigenstates). Their normalizations are:

$$\bar{u}_\sigma(\mathbf{p})u_{\sigma'}(\mathbf{p}) = +\delta_{\sigma\sigma'} , \quad (13)$$

$$\bar{v}_\sigma(\mathbf{p})v_{\sigma'}(\mathbf{p}) = -\delta_{\sigma\sigma'} , \quad (14)$$

$$\bar{u}_\sigma(\mathbf{p})v_{\sigma'}(\mathbf{p}) = 0 . \quad (15)$$

The bar over the 4-spinors signifies the Dirac conjugation.

Thus in this Section we have used the basis for charged particles in the $(S, 0) \oplus (0, S)$ representation (in general)

$$u_{+\sigma}(\mathbf{0}) = N(\sigma) \begin{pmatrix} 1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \quad u_{\sigma-1}(\mathbf{0}) = N(\sigma) \begin{pmatrix} 0 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}, \dots v_{-\sigma}(\mathbf{0}) = N(\sigma) \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix} \quad (16)$$

Sometimes, the normalization factor is convenient to choose $N(\sigma) = m^\sigma$ in order the rest spinors to vanish in the massless limit.

However, other constructs are possible in the $(1/2, 0) \oplus (0, 1/2)$ representation.

2 Majorana Spinors in the Momentum Representation.

During the 20th century various authors introduced *self/anti-self* charge-conjugate 4-spinors (including in the momentum rep-

resentation), see [Majorana, Bilenky, Ziino, Ahluwalia]. Later [Lounesto, Dvoeglazov1, Dvoeglazov2, Kirchbach] *etc* studied these spinors, they found corresponding dynamical equations, gauge transformations and other specific features of them. The definitions are:

$$C = e^{i\theta} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \mathcal{K} = -e^{i\theta} \gamma^2 \mathcal{K} \quad (17)$$

is the anti-linear operator of charge conjugation. \mathcal{K} is the complex conjugation operator. We define the *self/anti-self* charge-conjugate 4-spinors in the momentum space

$$C\lambda^{S,A}(\mathbf{p}) = \pm\lambda^{S,A}(\mathbf{p}), \quad (18)$$

$$C\rho^{S,A}(\mathbf{p}) = \pm\rho^{S,A}(\mathbf{p}). \quad (19)$$

Thus,

$$\lambda^{S,A}(p^\mu) = \begin{pmatrix} \pm i\Theta\phi_L^*(\mathbf{p}) \\ \phi_L(\mathbf{p}) \end{pmatrix}, \quad (20)$$

and

$$\rho^{S,A}(\mathbf{p}) = \begin{pmatrix} \phi_R(\mathbf{p}) \\ \mp i\Theta\phi_R^*(\mathbf{p}) \end{pmatrix}. \quad (21)$$

The Wigner matrix is

$$\Theta_{[1/2]} = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad (22)$$

and ϕ_L, ϕ_R can be boosted with $\Lambda_{L,R}$ matrices.¹

¹Such definitions of 4-spinors differ, of course, from the original Majorana definition in x-representation:

$$\nu(x) = \frac{1}{\sqrt{2}}(\Psi_D(x) + \Psi_D^c(x)), \quad (23)$$

The rest λ and ρ spinors are:

$$\lambda_{\uparrow}^S(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ i \\ 1 \\ 0 \end{pmatrix}, \lambda_{\downarrow}^S(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} -i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (24)$$

$$\lambda_{\uparrow}^A(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ -i \\ 1 \\ 0 \end{pmatrix}, \lambda_{\downarrow}^A(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} i \\ 0 \\ 0 \\ 1 \end{pmatrix}, \quad (25)$$

$$\rho_{\uparrow}^S(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -i \end{pmatrix}, \rho_{\downarrow}^S(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix}, \quad (26)$$

$$\rho_{\uparrow}^A(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ i \end{pmatrix}, \rho_{\downarrow}^A(\mathbf{0}) = \sqrt{\frac{m}{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \quad (27)$$

Thus, in this basis the explicit forms of the 4-spinors of the second kind $\lambda_{\uparrow\downarrow}^{S,A}(\mathbf{p})$ and $\rho_{\uparrow\downarrow}^{S,A}(\mathbf{p})$ are

$$\lambda_{\uparrow}^S(\mathbf{p}) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix} ip_l \\ i(p^- + m) \\ p^- + m \\ -p_r \end{pmatrix}, \lambda_{\downarrow}^S(\mathbf{p}) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix} -i(p^+ + m) \\ -ip_r \\ -p_l \\ (p^+ + m) \end{pmatrix} \quad (28)$$

$C\nu(x) = \nu(x)$ that represents the positive real C -parity field operator. However, the momentum-space Majorana-like spinors open various possibilities for description of neutral particles (with experimental consequences, see [Kirchbach]). For instance, "for imaginary C parities, the neutrino mass can drop out from the single β decay trace and reappear in $0\nu\beta\beta$, a curious and in principle experimentally testable signature for a non-trivial impact of Majorana framework in experiments with polarized sources."

$$\lambda_{\uparrow}^A(\mathbf{p}) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix} -ip_l \\ -i(p^- + m) \\ (p^- + m) \\ -p_r \end{pmatrix}, \lambda_{\downarrow}^A(\mathbf{p}) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix} i(p^+ + m) \\ ip_r \\ -p_l \\ (p^+ + m) \end{pmatrix} \quad (29)$$

$$\rho_{\uparrow}^S(\mathbf{p}) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix} p^+ + m \\ p_r \\ ip_l \\ -i(p^+ + m) \end{pmatrix}, \rho_{\downarrow}^S(\mathbf{p}) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix} p_l \\ (p^- + m) \\ i(p^- + m) \\ -ip_r \end{pmatrix} \quad (30)$$

$$\rho_{\uparrow}^A(\mathbf{p}) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix} p^+ + m \\ p_r \\ -ip_l \\ i(p^+ + m) \end{pmatrix}, \rho_{\downarrow}^A(\mathbf{p}) = \frac{1}{2\sqrt{E+m}} \begin{pmatrix} p_l \\ (p^- + m) \\ -i(p^- + m) \\ ip_r \end{pmatrix}. \quad (31)$$

As we showed λ and ρ 4-spinors are NOT the eigenspinors of the helicity. Moreover, λ and ρ are NOT the eigenspinors of the parity (in this representation $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} R$), as opposed to the Dirac case. The indices $\uparrow\downarrow$ should be referred to the chiral helicity quantum number introduced in the 60s, $\eta = -\gamma^5 h$. While

$$Pu_{\sigma}(\mathbf{p}) = +u_{\sigma}(\mathbf{p}), Pv_{\sigma}(\mathbf{p}) = -v_{\sigma}(\mathbf{p}), \quad (32)$$

we have

$$P\lambda^{S,A}(\mathbf{p}) = \rho^{A,S}(\mathbf{p}), P\rho^{S,A}(\mathbf{p}) = \lambda^{A,S}(\mathbf{p}), \quad (33)$$

for the Majorana-like momentum-space 4-spinors on the first quantization level. In this basis one has

$$\rho_{\uparrow}^S(\mathbf{p}) = -i\lambda_{\downarrow}^A(\mathbf{p}), \rho_{\downarrow}^S(\mathbf{p}) = +i\lambda_{\uparrow}^A(\mathbf{p}), \quad (34)$$

$$\rho_{\uparrow}^A(\mathbf{p}) = +i\lambda_{\downarrow}^S(\mathbf{p}), \rho_{\downarrow}^A(\mathbf{p}) = -i\lambda_{\uparrow}^S(\mathbf{p}). \quad (35)$$

The normalization of the spinors $\lambda_{\uparrow\downarrow}^{S,A}(\mathbf{p})$ and $\rho_{\uparrow\downarrow}^{S,A}(\mathbf{p})$ are the following ones:

$$\bar{\lambda}_{\uparrow}^S(\mathbf{p})\lambda_{\downarrow}^S(\mathbf{p}) = -im \quad , \quad \bar{\lambda}_{\downarrow}^S(\mathbf{p})\lambda_{\uparrow}^S(\mathbf{p}) = +im \quad , \quad (36)$$

$$\bar{\lambda}_{\uparrow}^A(\mathbf{p})\lambda_{\downarrow}^A(\mathbf{p}) = +im \quad , \quad \bar{\lambda}_{\downarrow}^A(\mathbf{p})\lambda_{\uparrow}^A(\mathbf{p}) = -im \quad , \quad (37)$$

$$\bar{\rho}_{\uparrow}^S(\mathbf{p})\rho_{\downarrow}^S(\mathbf{p}) = +im \quad , \quad \bar{\rho}_{\downarrow}^S(\mathbf{p})\rho_{\uparrow}^S(\mathbf{p}) = -im \quad , \quad (38)$$

$$\bar{\rho}_{\uparrow}^A(\mathbf{p})\rho_{\downarrow}^A(\mathbf{p}) = -im \quad , \quad \bar{\rho}_{\downarrow}^A(\mathbf{p})\rho_{\uparrow}^A(\mathbf{p}) = +im \quad . \quad (39)$$

All other conditions are equal to zero.

The dynamical coordinate-space equations are:

$$i\gamma^\mu\partial_\mu\lambda^S(x) - m\rho^A(x) = 0 \quad , \quad (40)$$

$$i\gamma^\mu\partial_\mu\rho^A(x) - m\lambda^S(x) = 0 \quad , \quad (41)$$

$$i\gamma^\mu\partial_\mu\lambda^A(x) + m\rho^S(x) = 0 \quad , \quad (42)$$

$$i\gamma^\mu\partial_\mu\rho^S(x) + m\lambda^A(x) = 0 \quad . \quad (43)$$

These are NOT the Dirac equation. However, they can be written in the 8-component form as follows:

$$[i\Gamma^\mu\partial_\mu - m]\Psi_{(+)}(x) = 0 \quad , \quad (44)$$

$$[i\Gamma^\mu\partial_\mu + m]\Psi_{(-)}(x) = 0 \quad , \quad (45)$$

with

$$\Psi_{(+)}(x) = \begin{pmatrix} \rho^A(x) \\ \lambda^S(x) \end{pmatrix} , \Psi_{(-)}(x) = \begin{pmatrix} \rho^S(x) \\ \lambda^A(x) \end{pmatrix} , \text{ and } \Gamma^\mu = \begin{pmatrix} 0 & \gamma^\mu \\ \gamma^\mu & 0 \end{pmatrix} \quad (46)$$

One can also re-write the equations into the two-component form. Similar formulations have been presented by M. Markov [Markov], and A. Barut and G. Ziino [Ziino]. The group-theoretical basis for such doubling has been given in the papers by Gelfand, Tsetlin and Sokolik [Gelfand].

The Lagrangian is

$$\begin{aligned} \mathcal{L} = \frac{i}{2} & \left[\bar{\lambda}^S \gamma^\mu \partial_\mu \lambda^S - (\partial_\mu \bar{\lambda}^S) \gamma^\mu \lambda^S + \bar{\rho}^A \gamma^\mu \partial_\mu \rho^A - (\partial_\mu \bar{\rho}^A) \gamma^\mu \rho^A + \right. \\ & \bar{\lambda}^A \gamma^\mu \partial_\mu \lambda^A - (\partial_\mu \bar{\lambda}^A) \gamma^\mu \lambda^A + \bar{\rho}^S \gamma^\mu \partial_\mu \rho^S - (\partial_\mu \bar{\rho}^S) \gamma^\mu \rho^S - \\ & \left. -m(\bar{\lambda}^S \rho^A + \bar{\lambda}^S \rho^A - \bar{\lambda}^S \rho^A - \bar{\lambda}^S \rho^A) \right] \end{aligned} \quad (47)$$

The connection with the Dirac spinors has been found. For instance,

$$\begin{pmatrix} \lambda_\uparrow^S(\mathbf{p}) \\ \lambda_\downarrow^S(\mathbf{p}) \\ \lambda_\uparrow^A(\mathbf{p}) \\ \lambda_\downarrow^A(\mathbf{p}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & i & -1 & i \\ -i & 1 & -i & -1 \\ 1 & -i & -1 & -i \\ i & 1 & i & -1 \end{pmatrix} \begin{pmatrix} u_{+1/2}(\mathbf{p}) \\ u_{-1/2}(\mathbf{p}) \\ v_{+1/2}(\mathbf{p}) \\ v_{-1/2}(\mathbf{p}) \end{pmatrix}. \quad (48)$$

See also ref. [Gelfand, Ziino].

The sets of λ spinors and of ρ spinors are claimed to be *bi-orthonormal* sets each in the mathematical sense [Ahluwalia], provided that overall phase factors of 2-spinors $\theta_1 + \theta_2 = 0$ or π . For instance, on the classical level $\bar{\lambda}_\uparrow^S \lambda_\downarrow^S = 2iN^2 \cos(\theta_1 + \theta_2)$.²

Few remarks have been given in the previous works:

- While in the massive case there are four λ -type spinors, two λ^S and two λ^A (the ρ spinors are connected by certain relations with the λ spinors for any spin case), in a massless case λ_\uparrow^S and λ_\uparrow^A identically vanish, provided that one takes into account that $\phi_L^{\pm 1/2}$ are eigenspinors of $\sigma \cdot \hat{\mathbf{n}}$, the 2×2 helicity operator.
- It was noted the possibility of the generalization of the concept of the Fock space, which leads to the “doubling” Fock space [Gelfand, Ziino].

²We used above $\theta_1 = \theta_2 = 0$.

It was shown [Dvoeglazov1] that the covariant derivative (and, hence, the interaction) can be introduced in this construct in the following way:

$$\partial_\mu \rightarrow \nabla_\mu = \partial_\mu - ig\mathbf{L}^5 A_\mu \quad , \quad (49)$$

where $\mathbf{L}^5 = \text{diag}(\gamma^5 \quad -\gamma^5)$, the 8×8 matrix. With respect to the transformations

$$\lambda'(x) \rightarrow (\cos \alpha - i\gamma^5 \sin \alpha)\lambda(x) \quad , \quad (50)$$

$$\bar{\lambda}'(x) \rightarrow \bar{\lambda}(x)(\cos \alpha - i\gamma^5 \sin \alpha) \quad , \quad (51)$$

$$\rho'(x) \rightarrow (\cos \alpha + i\gamma^5 \sin \alpha)\rho(x) \quad , \quad (52)$$

$$\bar{\rho}'(x) \rightarrow \bar{\rho}(x)(\cos \alpha + i\gamma^5 \sin \alpha) \quad (53)$$

the spinors retain their properties to be self/anti-self charge conjugate spinors and the proposed Lagrangian [Dvoeglazov1, p.1472] remains to be invariant. This tells us that while self/anti-self charge conjugate states have zero eigenvalues of the ordinary (scalar) charge operator but they can possess the axial charge (cf. with the discussion of [Ziino] and the old idea of R. E. Marshak).

In fact, from this consideration one can recover the Feynman-Gell-Mann equation (and its charge-conjugate equation). It is re-written in the two-component form

$$\begin{cases} \left[\pi_\mu^- \pi^{\mu-} - m^2 - \frac{g}{2} \sigma^{\mu\nu} F_{\mu\nu} \right] \chi(x) = 0 , \\ \left[\pi_\mu^+ \pi^{\mu+} - m^2 + \frac{g}{2} \tilde{\sigma}^{\mu\nu} F_{\mu\nu} \right] \phi(x) = 0 , \end{cases} \quad (54)$$

where already one has $\pi_\mu^\pm = i\partial_\mu \pm gA_\mu$, $\sigma^{0i} = -\tilde{\sigma}^{0i} = i\sigma^i$, $\sigma^{ij} = \tilde{\sigma}^{ij} = \epsilon_{ijk}\sigma^k$ and $\nu^{DL}(x) = \text{column}(\chi \quad \phi)$.

Next, because the transformations

$$\lambda'_S(\mathbf{p}) = \begin{pmatrix} \Xi & 0 \\ 0 & \Xi \end{pmatrix} \lambda_S(\mathbf{p}) \equiv \lambda_A^*(\mathbf{p}), \quad (55)$$

$$\lambda''_S(\mathbf{p}) = \begin{pmatrix} i\Xi & 0 \\ 0 & -i\Xi \end{pmatrix} \lambda_S(\mathbf{p}) \equiv -i\lambda_S^*(\mathbf{p}), \quad (56)$$

$$\lambda'''_S(\mathbf{p}) = \begin{pmatrix} 0 & i\Xi \\ i\Xi & 0 \end{pmatrix} \lambda_S(\mathbf{p}) \equiv i\gamma^0\lambda_A^*(\mathbf{p}), \quad (57)$$

$$\lambda_S^{IV}(\mathbf{p}) = \begin{pmatrix} 0 & \Xi \\ -\Xi & 0 \end{pmatrix} \lambda_S(\mathbf{p}) \equiv \gamma^0\lambda_S^*(\mathbf{p}) \quad (58)$$

with the 2×2 matrix Ξ defined as (ϕ is the azimuthal angle related with $\mathbf{p} \rightarrow \mathbf{0}$)

$$\Xi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}, \quad \Xi \Lambda_{R,L}(\mathbf{p} \leftarrow \mathbf{0}) \Xi^{-1} = \Lambda_{R,L}^*(\mathbf{p} \leftarrow \mathbf{0}), \quad (59)$$

and corresponding transformations for λ^A do *not* change the properties of bispinors to be in the self/anti-self charge conjugate spaces, the Majorana-like field operator ($b^\dagger \equiv a^\dagger$) admits additional phase (and, in general, normalization) transformations:

$$\nu^{ML\prime}(x^\mu) = [c_0 + i(\boldsymbol{\tau} \cdot \mathbf{c})] \nu^{ML\dagger}(x^\mu), \quad (60)$$

where c_α are arbitrary parameters. The τ matrices are defined over the field of 2×2 matrices and the Hermitian conjugation operation is assumed to act on the c -numbers as the complex conjugation. One can parametrize $c_0 = \cos \phi$ and $\mathbf{c} = \mathbf{n} \sin \phi$ and, thus, define the $SU(2)$ group of phase transformations. One can select the Lagrangian which is composed from the both field operators (with λ spinors and ρ spinors) and which remains to be invariant with respect to this kind of transformations. The conclusion is: it is permitted a non-Abelian construct which

is based on the spinors of the Lorentz group only (cf. with the old ideas of T. W. Kibble and R. Utiyama) . This is not surprising because both the $SU(2)$ group and $U(1)$ group are the sub-groups of the extended Poincaré group (cf. [Ryder]).

The Dirac-like and Majorana-like field operators can be built from both $\lambda^{S,A}(\mathbf{p})$ and $\rho^{S,A}(\mathbf{p})$, or their combinations. For instance,

$$\begin{aligned} \Psi(x^\mu) \equiv & \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_p} \sum_{\eta} [\lambda_{\eta}^S(\mathbf{p}) a_{\eta}(\mathbf{p}) \exp(-ip \cdot x) + \\ & + \lambda_{\eta}^A(\mathbf{p}) b_{\eta}^{\dagger}(\mathbf{p}) \exp(+ip \cdot x)] . \end{aligned} \quad (61)$$

The anticommutation relations are the following ones (due to the *bi-orthonormality*):

$$[a_{\eta'}(\mathbf{p}'), a_{\eta}^{\dagger}(\mathbf{p})]_{\pm} = (2\pi)^3 2E_p \delta(\mathbf{p} - \mathbf{p}') \delta_{\eta, -\eta'} \quad (62)$$

and

$$[b_{\eta'}(\mathbf{p}'), b_{\eta}^{\dagger}(\mathbf{p})]_{\pm} = (2\pi)^3 2E_p \delta(\mathbf{p} - \mathbf{p}') \delta_{\eta, -\eta'} \quad (63)$$

Other (anti)commutators are equal to zero: $([a_{\eta'}(\mathbf{p}'), b_{\eta}^{\dagger}(\mathbf{p})] = 0)$.

In the Fock space operations of the charge conjugation and space inversions can be defined through unitary operators such that:

$$U_{[1/2]}^c \Psi(x^\mu) (U_{[1/2]}^c)^{-1} = \mathcal{C}_{[1/2]} \Psi_{[1/2]}^{\dagger}(x^\mu), \quad (64)$$

$$U_{[1/2]}^s \Psi(x^\mu) (U_{[1/2]}^s)^{-1} = \gamma^0 \Psi(x'^{\mu}), \quad (65)$$

the time reversal operation, through *an antiunitary* operator³

$$\left[V_{[1/2]}^T \Psi(x^\mu) (V_{[1/2]}^T)^{-1} \right]^{\dagger} = S(T) \Psi^{\dagger}(x''^{\mu}) \quad , \quad (66)$$

³Let us remind that the operator of hermitian conjugation does not act on *c*-numbers on the left side of the equation (66). This fact is connected with the properties of an antiunitary operator: $\left[V^T \lambda A (V^T)^{-1} \right]^{\dagger} = \left[\lambda^* V^T A (V^T)^{-1} \right]^{\dagger} = \lambda \left[V^T A^{\dagger} (V^T)^{-1} \right]$.

with $x'^{\mu} \equiv (x^0, -\mathbf{x})$ and $x''^{\mu} = (-x^0, \mathbf{x})$. We further assume the vacuum state to be assigned an even P - and C -eigenvalue and, then, proceed as in ref. [Itsykson].

As a result we have the following properties of creation (annihilation) operators in the Fock space:

$$\begin{aligned} U_{[1/2]}^s a_{\uparrow}(\mathbf{p})(U_{[1/2]}^s)^{-1} &= -ia_{\downarrow}(-\mathbf{p}), \\ U_{[1/2]}^s a_{\downarrow}(\mathbf{p})(U_{[1/2]}^s)^{-1} &= +ia_{\uparrow}(-\mathbf{p}) \end{aligned} \quad (67)$$

$$\begin{aligned} U_{[1/2]}^s b_{\uparrow}^{\dagger}(\mathbf{p})(U_{[1/2]}^s)^{-1} &= +ib_{\downarrow}^{\dagger}(-\mathbf{p}), \\ U_{[1/2]}^s b_{\downarrow}^{\dagger}(\mathbf{p})(U_{[1/2]}^s)^{-1} &= -ib_{\uparrow}^{\dagger}(-\mathbf{p}), \end{aligned} \quad (68)$$

what signifies that the states created by the operators $a^{\dagger}(\mathbf{p})$ and $b^{\dagger}(\mathbf{p})$ have very different properties with respect to the space inversion operation, comparing with Dirac states (the case also regarded in [Ziino]):

$$U_{[1/2]}^s |\mathbf{p}, \uparrow>^+ = +i|-\mathbf{p}, \downarrow>^+, U_{[1/2]}^s |\mathbf{p}, \uparrow>^- = +i|-\mathbf{p}, \downarrow>^- \quad (69)$$

$$U_{[1/2]}^s |\mathbf{p}, \downarrow>^+ = -i|-\mathbf{p}, \uparrow>^+, U_{[1/2]}^s |\mathbf{p}, \downarrow>^- = -i|-\mathbf{p}, \uparrow>^-. \quad (70)$$

For the charge conjugation operation in the Fock space we have two physically different possibilities. The first one, *e.g.*,

$$U_{[1/2]}^c a_{\uparrow}(\mathbf{p})(U_{[1/2]}^c)^{-1} = +b_{\uparrow}(\mathbf{p}), U_{[1/2]}^c a_{\downarrow}(\mathbf{p})(U_{[1/2]}^c)^{-1} = +b_{\downarrow}(\mathbf{p}), \quad (71)$$

$$U_{[1/2]}^c b_{\uparrow}^{\dagger}(\mathbf{p})(U_{[1/2]}^c)^{-1} = -a_{\uparrow}^{\dagger}(\mathbf{p}), U_{[1/2]}^c b_{\downarrow}^{\dagger}(\mathbf{p})(U_{[1/2]}^c)^{-1} = -a_{\downarrow}^{\dagger}(\mathbf{p}), \quad (72)$$

in fact, has some similarities with the Dirac construct. However, the action of this operator on the physical states are

$$U_{[1/2]}^c |\mathbf{p}, \uparrow>^+ = +|\mathbf{p}, \uparrow>^-, U_{[1/2]}^c |\mathbf{p}, \downarrow>^+ = +|\mathbf{p}, \downarrow>^-, \quad (73)$$

$$U_{[1/2]}^c |\mathbf{p}, \uparrow>^- = -|\mathbf{p}, \uparrow>^+, U_{[1/2]}^c |\mathbf{p}, \downarrow>^- = -|\mathbf{p}, \downarrow>^+. \quad (74)$$

But, one can also construct the charge conjugation operator in the Fock space which acts, *e.g.*, in the following manner:

$$\widetilde{U}_{[1/2]}^c a_{\uparrow}(\mathbf{p})(\widetilde{U}_{[1/2]}^c)^{-1} = -b_{\downarrow}(\mathbf{p}), \quad \widetilde{U}_{[1/2]}^c a_{\downarrow}(\mathbf{p})(\widetilde{U}_{[1/2]}^c)^{-1} = -b_{\uparrow}(\mathbf{p}), \quad (75)$$

$$\widetilde{U}_{[1/2]}^c b_{\uparrow}^{\dagger}(\mathbf{p})(\widetilde{U}_{[1/2]}^c)^{-1} = +a_{\downarrow}^{\dagger}(\mathbf{p}), \quad \widetilde{U}_{[1/2]}^c b_{\downarrow}^{\dagger}(\mathbf{p})(\widetilde{U}_{[1/2]}^c)^{-1} = +a_{\uparrow}^{\dagger}(\mathbf{p}), \quad (76)$$

and, therefore,

$$\widetilde{U}_{[1/2]}^c |\mathbf{p}, \uparrow>^+ = -|\mathbf{p}, \downarrow>^-, \quad \widetilde{U}_{[1/2]}^c |\mathbf{p}, \downarrow>^+ = -|\mathbf{p}, \uparrow>^-, \quad (77)$$

$$\widetilde{U}_{[1/2]}^c |\mathbf{p}, \uparrow>^- = +|\mathbf{p}, \downarrow>^+, \quad \widetilde{U}_{[1/2]}^c |\mathbf{p}, \downarrow>^- = +|\mathbf{p}, \uparrow>^+ \quad (78)$$

Investigations of several important cases, which are different from the above ones, are required a separate paper to. Next, it is possible a situation when the operators of the space inversion and charge conjugation commute each other in the Fock space [Foldy]. For instance,

$$U_{[1/2]}^c U_{[1/2]}^s |\mathbf{p}, \uparrow>^+ = +i U_{[1/2]}^c |-\mathbf{p}, \downarrow>^+ = +i |-\mathbf{p}, \downarrow>^- \quad (79)$$

$$U_{[1/2]}^s U_{[1/2]}^c |\mathbf{p}, \uparrow>^+ = U_{[1/2]}^s |\mathbf{p}, \uparrow>^- = +i |-\mathbf{p}, \downarrow>^- . \quad (80)$$

The second choice of the charge conjugation operator answers for the case when the $\widetilde{U}_{[1/2]}^c$ and $U_{[1/2]}^s$ operations anticommute:

$$\widetilde{U}_{[1/2]}^c U_{[1/2]}^s |\mathbf{p}, \uparrow>^+ = +i \widetilde{U}_{[1/2]}^c |-\mathbf{p}, \downarrow>^+ = -i |-\mathbf{p}, \uparrow>^- \quad (81)$$

$$U_{[1/2]}^s \widetilde{U}_{[1/2]}^c |\mathbf{p}, \uparrow>^+ = -U_{[1/2]}^s |\mathbf{p}, \downarrow>^- = +i |-\mathbf{p}, \uparrow>^- . \quad (82)$$

Next, one can compose states which would have somewhat similar properties to those which we have become accustomed. The states $|\mathbf{p}, \uparrow>^+ \pm i |\mathbf{p}, \downarrow>^+$ answer for positive (negative)

parity, respectively. But, what is important, *the antiparticle states* (moving backward in time) have the same properties with respect to the operation of space inversion as the corresponding *particle states* (as opposed to $j = 1/2$ Dirac particles). The states which are eigenstates of the charge conjugation operator in the Fock space are

$$U_{[1/2]}^c (|\mathbf{p}, \uparrow >^+ \pm i |\mathbf{p}, \uparrow >^-) = \mp i (|\mathbf{p}, \uparrow >^+ \pm i |\mathbf{p}, \uparrow >^-) . \quad (83)$$

There is no any simultaneous set of states which would be eigenstates of the operator of the space inversion and of the charge conjugation $U_{[1/2]}^c$.

Finally, the time reversal *anti-unitary* operator in the Fock space should be defined in such a way that the formalism to be compatible with the *CPT* theorem. If we wish the Dirac states to transform as $V(T)|\mathbf{p}, \pm 1/2 > = \pm |-\mathbf{p}, \mp 1/2 >$ we have to choose (within a phase factor), ref. [Itsykson]:

$$S(T) = \begin{pmatrix} \Theta_{[1/2]} & 0 \\ 0 & \Theta_{[1/2]} \end{pmatrix} . \quad (84)$$

Thus, in the first relevant case we obtain for the $\Psi(x^\mu)$ field, Eq. (61):

$$V^T a_\uparrow^\dagger(\mathbf{p})(V^T)^{-1} = a_\downarrow^\dagger(-\mathbf{p}), V^T a_\downarrow^\dagger(\mathbf{p})(V^T)^{-1} = -a_\uparrow^\dagger(-\mathbf{p}) \quad (85)$$

$$V^T b_\uparrow(\mathbf{p})(V^T)^{-1} = b_\downarrow(-\mathbf{p}), V^T b_\downarrow(\mathbf{p})(V^T)^{-1} = -b_\uparrow(-\mathbf{p}). \quad (86)$$

Thus, this construct has very different properties with respect to C, P and T comparing with the Dirac construct.

But, at least for mathematicians, the dependence of the physical results on the choice of the basis is a bit strange thing. Somewhat similar things have been presented in [Dvoeglazov3] when compared the Dirac-like

constructs in the parity and helicity bases. It was shown that the helicity eigenstates $(\sigma \cdot \mathbf{n}) \otimes I$ are NOT the parity eigenstates (and the S_3 eigenstates), and vice versa, in the helicity basis (cf. with [Berestetskii, Lifshitz, Pitaevskii]), while they obey the same Dirac equation. The bases are connected by the unitary transformation. And, the both sets of 4-spinors form the complete system in a mathematical sense.

3 The Spin 1.

3.1 Maxwell Equations as Quantum Equations.

In refs. [Gersten, Dvoeglazov4] the Maxwell-like equations have been derived⁴ from the Klein-Gordon equation. Here they are:

$$\nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} + \nabla \text{Im} \chi, \quad (87)$$

$$\nabla \times \mathbf{B} = \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} + \nabla \text{Re} \chi, \quad (88)$$

$$\nabla \cdot \mathbf{E} = -\frac{1}{c} \frac{\partial}{\partial t} \text{Re} \chi, \quad (89)$$

$$\nabla \cdot \mathbf{B} = \frac{1}{c} \frac{\partial}{\partial t} \text{Im} \chi. \quad (90)$$

Of course, similar equations can be obtained in the massive case $m \neq 0$, i.e., within the Proca-like theory. We should then consider

$$(E^2 - c^2 \mathbf{p}^2 - m^2 c^4) \Psi^{(3)} = 0. \quad (91)$$

⁴I call them "Maxwell-like" because an additional gradient of a scalar field χ can be introduced therein.

In the spin-1/2 case the equation (91) can be written for the two-component spinor ($c = \hbar = 1$)

$$(EI^{(2)} - \sigma \cdot \mathbf{p})(EI^{(2)} + \sigma \cdot \mathbf{p})\Psi^{(2)} = m^2\Psi^{(2)}, \quad (92)$$

or, in the 4-component form

$$[i\gamma_\mu\partial_\mu + m_1 + m_2\gamma^5]\Psi^{(4)} = 0. \quad (93)$$

In the spin-1 case we have

$$(EI^{(3)} - \mathbf{S} \cdot \mathbf{p})(EI^{(3)} + \mathbf{S} \cdot \mathbf{p})\Psi^{(3)} - \mathbf{p}(\mathbf{p} \cdot \Psi^{(3)}) = m^2\Psi^{(3)}. \quad (94)$$

These lead to (87-90), when $m = 0$ provided that the $\Psi^{(3)}$ is chosen as a superposition of a vector (the electric field) and an axial vector (the magnetic field).⁵ When $\chi = 0$ we recover the common-used Maxwell equations.

Otherwise, we can start with ($c = \hbar = 1$)⁶

$$\frac{\partial \mathbf{E}}{\partial t} = \text{curl} \mathbf{B}, \quad \frac{\partial \mathbf{B}}{\partial t} = -\text{curl} \mathbf{E}. \quad (95)$$

Then,

$$\frac{\partial(\mathbf{E} + i\mathbf{B})}{\partial t} - \text{curl}(\mathbf{B} - i\mathbf{E}) = 0, \quad (96)$$

$$\frac{\partial(\mathbf{E} - i\mathbf{B})}{\partial t} - \text{curl}(\mathbf{B} + i\mathbf{E}) = 0. \quad (97)$$

In the component form:

$$\frac{\partial(\mathbf{E} + i\mathbf{B})^i}{\partial t} + i\epsilon^{ijk}\partial_j(\mathbf{E} + i\mathbf{B})^k = 0, \quad (98)$$

$$\frac{\partial(\mathbf{E} - i\mathbf{B})^i}{\partial t} - i\epsilon^{ijk}\partial_j(\mathbf{E} - i\mathbf{B})^k = 0. \quad (99)$$

⁵We can continue writing down equations for higher spins in a similar fashion.

⁶The question of both explicite and implicate dependences of the fields on the time (and, hence, the "whole-partial derivative") has been studied in [Brownstein, Dvoeglazov5].

Since the spin-1 matrices can be presented in the form: $(\mathbf{S}^i)^{jk} = -i\epsilon^{ijk}$, we have

$$\frac{\partial(\mathbf{E} + i\mathbf{B})^i}{\partial t} + (\mathbf{S} \cdot \nabla)^{ik}(\mathbf{E} + i\mathbf{B})^k = 0, \quad (100)$$

$$\frac{\partial(\mathbf{E} - i\mathbf{B})^i}{\partial t} - (\mathbf{S} \cdot \nabla)^{ik}(\mathbf{E} - i\mathbf{B})^k = 0. \quad (101)$$

Finally, on using that $\hat{\mathbf{p}} = -i\hbar\nabla$ we have

$$i\frac{\partial\phi}{\partial t} = (\mathbf{S} \cdot \hat{\mathbf{p}})\phi, \quad i\frac{\partial\xi}{\partial t} = -(\mathbf{S} \cdot \hat{\mathbf{p}})\xi. \quad (102)$$

In the following we show that these equations can also be considered as the massless limit of the Weinberg $S = 1$ quantum-field equation.

Meanwhile, we can calculate the determinants of the above equations, $\text{Det}[E \mp (\mathbf{S} \cdot \mathbf{p})] = 0$, and we can find that we have both the causal $E = \pm|\mathbf{p}|$ and acausal $E = 0$ solutions.⁷ These results will be useful in analyzing the spin-1 quantum-field theory below.

3.2 The Weinberg $2(2S + 1)$ Theory for Spin-1

It is based on the following postulates [Wigner, Weinberg]:

- The fields transform according to the formula:

$$U[\Lambda, a]\Psi_n(x)U^{-1}[\Lambda, a] = \sum_m D_{nm}[\Lambda^{-1}]\Psi_m(\Lambda x + a), \quad (103)$$

where $D_{nm}[\Lambda]$ is some representation of Λ ; $x^\mu \rightarrow \Lambda^\mu{}_\nu x^\nu + a^\mu$, and $U[\Lambda, a]$ is a unitary operator.

⁷The possible interpretation of the $E = 0$ solutions are the stationary fields.

- For $(x - y)$ spacelike one has

$$[\Psi_n(x), \Psi_m(y)]_{\pm} = 0 \quad (104)$$

for fermion and boson fields, respectively.

- The interaction Hamiltonian density is said by S. Weinberg to be a scalar, and it is constructed out of the creation and annihilation operators for the free particles described by the free Hamiltonian H_0 .
- The S -matrix is constructed as an integral of the T -ordering product of the interaction Hamiltonians by the Dyson's formula.

In this talk we shall be mainly interested in the free-field theory. Weinberg wrote: “In order to discuss theories with parity conservation it is convenient to use $2(2S + 1)$ -component fields, like the Dirac field. These do obey field equations, which can be derived as... consequences of (103,104).”⁸ In such a way he proceeds to form the $2(2S + 1)$ -component object

$$\Psi = \begin{pmatrix} \Phi_{\sigma} \\ \Xi_{\sigma} \end{pmatrix}$$

transforming according to the Wigner rules. They are the following ones (see also above, Eqs. (5,6)):

$$\Phi_{\sigma}(\mathbf{p}) = \exp(+\Theta \hat{\mathbf{p}} \cdot \mathbf{S}) \Phi_{\sigma}(\mathbf{0}), \quad (105)$$

$$\Xi_{\sigma}(\mathbf{p}) = \exp(-\Theta \hat{\mathbf{p}} \cdot \mathbf{S}) \Xi_{\sigma}(\mathbf{0}) \quad (106)$$

from the zero-momentum frame. Θ is the boost parameter, $\tanh \Theta = |\mathbf{p}|/E$, $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$, \mathbf{p} is the 3-momentum of the particle, \mathbf{S} is the angular momentum operator. For a given representation the matrices \mathbf{S} can be constructed. In the Dirac case (the

⁸In the $(2S + 1)$ formalism fields obey only the Klein-Gordon equation, according to the Weinberg wisdom.

$(1/2, 0) \oplus (0, 1/2)$ representation) $\mathbf{S} = \sigma/2$; in the $S = 1$ case (the $(1, 0) \oplus (0, 1)$ representation) we can choose $(S_i)_{jk} = -i\epsilon_{ijk}$, etc. Hence, we can explicitly calculate (105,106).

The task is now to obtain relativistic equations for higher spins. Weinberg uses the following procedure. Firstly, he defined the scalar matrix

$$\Pi_{\sigma'\sigma}^{(s)}(q) = (-)^{2s} t_{\sigma'\sigma}^{\mu_1\mu_2\cdots\mu_{2s}} q_{\mu_1} q_{\mu_2} \cdots q_{\mu_{2s}} \quad (107)$$

for the $(S, 0)$ representation of the Lorentz group ($q_\mu q_\mu = -m^2$), with the tensor t being defined by [Weinberg, Eqs. (A4-A5)]. Hence,

$$D^{(s)}[\Lambda] \Pi^{(s)}(q) D^{(s)\dagger}[\Lambda] = \Pi^{(s)}(\Lambda q) \quad (108)$$

Since at rest we have $[\mathbf{S}^{(s)}, \Pi^{(s)}(m)] = 0$, then according to the Schur's lemma $\Pi_{\sigma\sigma'}^{(s)}(m) = m^{2s} \delta_{\sigma\sigma'}$. After the substitution of $D^{(s)}[\Lambda]$ in Eq. (108) one has

$$\Pi^{(s)}(q) = m^{2s} \exp(2\Theta \hat{\mathbf{q}} \cdot \mathbf{S}^{(s)}) . \quad (109)$$

One can construct the analogous matrix for the $(0, S)$ representation by the same procedure:

$$\bar{\Pi}^{(s)}(q) = m^{2s} \exp(-2\Theta \hat{\mathbf{q}} \cdot \mathbf{S}^{(s)}) . \quad (110)$$

Finally, by the direct verification one has in the coordinate representation

$$\bar{\Pi}_{\sigma\sigma'}(-i\partial)\Phi_{\sigma'} = m^{2s}\Xi_{\sigma} , \quad (111)$$

$$\Pi_{\sigma\sigma'}(-i\partial)\Xi_{\sigma'} = m^{2s}\Phi_{\sigma} , \quad (112)$$

provided that $\Phi_{\sigma}(\mathbf{0})$ and $\Xi_{\sigma}(\mathbf{0})$ are indistinguishable.⁹

⁹Later, this fact has been incorporated in the Ryder book [Ryder]. Truly speaking, this is an additional postulate. It is possible that the zero-momentum-frame $2(2S+1)$ -component objects (the 4-spinor in the $(1/2, 0) \oplus (0, 1/2)$ representation, the bivector in the $(1, 0) \oplus (0, 1)$ representation, etc.) are connected by an arbitrary phase factor [Dvoeglazov6].

As a result one has

$$[\gamma^{\mu_1\mu_2\cdots\mu_{2s}}\partial_{\mu_1}\partial_{\mu_2}\cdots\partial_{\mu_{2s}} + m^{2s}]\Psi(x) = 0, \quad (113)$$

with the Barut-Muzinich-Williams covariantly-defined matrices [BMW, Sankaranarayanan,Good]. For the spin-1 they are:

$$\gamma_{44} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_{i4} = \gamma_{4i} = \begin{pmatrix} 0 & iS_i \\ -iS_i & 0 \end{pmatrix}, \quad (114)$$

$$\gamma^{ij} = \begin{pmatrix} 0 & \delta_{ij} - S_i S_j - S_j S_i \\ \delta_{ij} - S_i S_j - S_j S_i & 0 \end{pmatrix}. \quad (115)$$

Later Sankaranarayanan and Good considered another version of this theory [Sankaranarayanan,Good] (see also [Ahluwalia2]). For the $S = 1$ case they introduced the Weaver-Hammer-Good sign operator, ref. [Weaver], $m^2 \rightarrow m^2 (i\partial/\partial t)/E$, which led to the different parity properties of an antiparticle with respect to a *boson* particle. Next, Tucker and Hammer *et al* [Tucker,Hammer] introduced another higher-spin equations. In the spin-1 case it is:

$$[\gamma_{\mu\nu}\partial_\mu\partial_\nu + \partial_\mu\partial_\mu - 2m^2]\Psi^{(s=1)} = 0 \quad (116)$$

(Euclidean metric is now used). In fact, they added the Klein-Gordon equation to the Weinberg equation. One can add the Klein-Gordon equation with arbitrary multiple factor to the Weinberg equation. So, we can study the generalized Weinberg-Tucker-Hammer equation ($S = 1$), which is written ($p_\mu = -i\partial/\partial x^\mu$):

$$[\gamma_{\alpha\beta}p_\alpha p_\beta + Ap_\alpha p_\alpha + Bm^2]\Psi = 0. \quad (117)$$

It has solutions with relativistic dispersion relations $E^2 - \mathbf{p}^2 = m^2$, ($c = \hbar = 1$) provided that

$$\frac{B}{A+1} = 1, \quad \text{or} \quad \frac{B}{A-1} = 1. \quad (118)$$

This can be proven by considering the algebraic equation $Det[\gamma_{\alpha\beta}p_\alpha p_\beta + Ap_\alpha p_\alpha + Bm^2] = 0$. It is of the 12th order in p_μ . Solving it with respect to energy one obtains the conditions (118). **Unlike the Maxwell equations there are NO any $E = 0$ solutions.**

The solutions in the momentum representation have been explicitly presented by [Ahluwalia2]:

$$u_{+1}(\mathbf{p}) = \begin{pmatrix} m + [(2p_z^2 + p_+p_-)/2(E + m)] \\ p_z p_+ / \sqrt{2}(E + m) \\ p_+^2 / 2(E + m) \\ p_z \\ p_+ / \sqrt{2} \\ 0 \end{pmatrix}, \quad (119)$$

$$u_0(\mathbf{p}) = \begin{pmatrix} p_z p_- / \sqrt{2}(E + m) \\ m + [p_+ p_- / (E + m)] \\ -p_z p_+ / \sqrt{2}(E + m) \\ p_- / \sqrt{2} \\ 0 \\ p_+ / \sqrt{2} \end{pmatrix}, \quad (120)$$

$$u_{-1}(\mathbf{p}) = \begin{pmatrix} p_-^2 / 2(E + m) \\ -p_z p_- / \sqrt{2}(E + m) \\ m + [(2p_z^2 + p_+ p_-)/2(E + m)] \\ 0 \\ p_- / \sqrt{2} \\ -p_z \end{pmatrix}, \quad (121)$$

and

$$v_\sigma(\mathbf{p}) = \gamma_5 u_\sigma(\mathbf{p}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} U_\sigma(\mathbf{p}) \quad (122)$$

in the standard representation of $\gamma_{\mu\nu}$ matrices. If the 6-component $v(\mathbf{p})$ are defined in such way, we inevitably would get the ad-

ditional energy-sign operator [Weaver, Sankaranarayanan, Good] $\epsilon = i\partial_t/E = \pm 1$ in the dynamical equation, and the different parities of the corresponding boson and antiboson, $\hat{P}u_\sigma(\mathbf{p}) = +u_\sigma(\mathbf{p})$ and $\hat{P}v_\sigma(\mathbf{p}) = -v_\sigma(\mathbf{p})$.

4 The Construction of Field Operators.

The method for constructions of field operators has been given in [Bogoliubov, Shirkov]:¹⁰

$$\phi(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \tilde{\phi}(k). \quad (123)$$

From the Klein-Gordon equation we know:

$$(k^2 - m^2)\tilde{\phi}(k) = 0. \quad (124)$$

Thus,

$$\tilde{\phi}(k) = \delta(k^2 - m^2)\phi(k). \quad (125)$$

Next,

$$\begin{aligned} \phi(x) &= \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2)(\theta(k_0) + \theta(-k_0))\phi(k) = \\ &= \frac{1}{(2\pi)^{3/2}} \int dk [e^{ikx} \delta(k^2 - m^2)\phi^+(k) + e^{-ikx} \delta(k^2 - m^2)\phi^-(k)], \end{aligned} \quad (126)$$

where

$$\phi^+(k) = \theta(k_0)\phi(k), \text{ and } \phi^-(k) = \theta(k_0)\phi(-k). \quad (127)$$

¹⁰In this book a bit different notation for positive- (negative-) energy solutions has been used comparing with the general accepted one.

$$\phi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{2E_k} e^{+ikx} \phi^+(k), \quad (128)$$

$$\phi^-(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{2E_k} e^{-ikx} \phi^-(k). \quad (129)$$

In the spinor case (the $(1/2, 0) \oplus (0, 1/2)$ representation space) we have more components. Instead of the equation (124) we have

$$(\hat{k} + m)\psi(k)|_{k^2=m^2} = 0. \quad (130)$$

However, again

$$\psi(x) = \frac{1}{(2\pi)^{3/2}} \int dk e^{ikx} \delta(k^2 - m^2) (\theta(k_0) + \theta(-k_0)) \psi(k), \quad (131)$$

and

$$\psi(x) = \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2E_k} [e^{ikx} \theta(k_0) \psi(k) + e^{-ikx} \theta(k_0) \psi(-k)], \quad (132)$$

where $k_0 = E = \sqrt{\mathbf{k}^2 + m^2}$ is positive in this case. Hence:

$$(\hat{k} + m)\psi^+(\mathbf{k}) = 0, \quad (-\hat{k} + m)\psi^-(\mathbf{k}) = 0. \quad (133)$$

Everything is OK? However, please note that the momentum-space Dirac equations $(\hat{k} - m)u = 0$, $(\hat{k} + m)v = 0$ have solutions $k_0 = \pm\sqrt{\mathbf{k}^2 + m^2}$, both for u - and v -spinors. This can be checked by calculating the determinants. Usually, one chooses $k_0 = E = \sqrt{\mathbf{k}^2 + m^2}$ in the u - and in the v -. This is because on the classical level (better to say, on the first quantization level) the negative-energy u - can be transformed in the positive-energy v -, and vice versa. This is not precisely so, if

we go to the secondary quantization level. The introduction of creation/annihilation noncommutating operators gives us more possibilities in constructing generalized theory even on the basis of the Dirac equation.

Various-type field operators are possible in the $(1/2, 1/2)$ representation. During the calculations below we have to present $1 = \theta(k_0) + \theta(-k_0)$ (as previously) in order to get positive- and negative-frequency parts.

$$\begin{aligned}
A_\mu(x) &= \frac{1}{(2\pi)^3} \int d^4k \delta(k^2 - m^2) e^{+ik \cdot x} A_\mu(k) = \\
&= \frac{1}{(2\pi)^3} \sum_\lambda \int d^4k \delta(k_0^2 - E_k^2) e^{+ik \cdot x} \epsilon_\mu(k, \lambda) a_\lambda(k) = \\
&= \frac{1}{(2\pi)^3} \int \frac{d^4k}{2E} [\delta(k_0 - E_k) + \delta(k_0 + E_k)] [\theta(k_0) + \theta(-k_0)] \\
&\quad e^{+ik \cdot x} A_\mu(k) = \frac{1}{(2\pi)^3} \int \frac{d^4k}{2E} [\delta(k_0 - E_k) + \delta(k_0 + E_k)] \\
&\quad [\theta(k_0) A_\mu(k) e^{+ik \cdot x} + \theta(k_0) A_\mu(-k) e^{-ik \cdot x}] = \tag{134} \\
&= \frac{1}{(2\pi)^3} \int \frac{d^3\mathbf{k}}{2E_k} \theta(k_0) [A_\mu(k) e^{+ik \cdot x} + A_\mu(-k) e^{-ik \cdot x}] = \\
&= \frac{1}{(2\pi)^3} \sum_\lambda \int \frac{d^3\mathbf{k}}{2E_k} [\epsilon_\mu(k, \lambda) a_\lambda(k) e^{+ik \cdot x} + \epsilon_\mu(-k, \lambda) a_\lambda(-k) e^{-ik \cdot x}].
\end{aligned}$$

In general, due to theorems for integrals and for distributions the presentation $1 = \theta(k_0) + \theta(-k_0)$ is possible because we use this in the integrand. However, remember, that we have the $k_0 = E = 0$ solution of the Maxwell equations.¹¹ Moreover, it has the experimental confirmation (for instance, the stationary mag-

¹¹Of course, the same procedure can be applied in the construction of the quantum field operator for $F_{\mu\nu}$.

netic field $\text{curl}\mathbf{B} = 0$). Meanwhile the *theta* function is NOT defined in $k_0 = 0$. Do we not loose this solution in the above construction of the quantum field operator? Mathematicians did not answer me in a straightforward way.

Moreover, we should transform the second part to $\epsilon_\mu^*(k, \lambda)b_\lambda^\dagger(k)$ as usual. In such a way we obtain the charge-conjugate states.¹² Of course, one can try to get P -conjugates or CP -conjugate states too.

In the Dirac case we should assume the following relation in the field operator:

$$\sum_\lambda v_\lambda(k)b_\lambda^\dagger(k) = \sum_\lambda u_\lambda(-k)a_\lambda(-k). \quad (135)$$

We know that [Ryder, Itsykson]

$$\bar{u}_\mu(k)u_\lambda(k) = +m\delta_{\mu\lambda}, \quad (136)$$

$$\bar{u}_\mu(k)u_\lambda(-k) = 0, \quad (137)$$

$$\bar{v}_\mu(k)v_\lambda(k) = -m\delta_{\mu\lambda}, \quad (138)$$

$$\bar{v}_\mu(k)u_\lambda(k) = 0, \quad (139)$$

but we need $\Lambda_{\mu\lambda}(k) = \bar{v}_\mu(k)u_\lambda(-k)$. By direct calculations, we find

$$-mb_\mu^\dagger(k) = \sum_\nu \Lambda_{\mu\nu}(k)a_\nu(-k). \quad (140)$$

Hence, $\Lambda_{\mu\lambda} = -im(\sigma \cdot \mathbf{n})_{\mu\lambda}$ and

$$b_\mu^\dagger(k) = i(\sigma \cdot \mathbf{n})_{\mu\lambda}a_\lambda(-k). \quad (141)$$

Multiplying (135) by $\bar{u}_\mu(-k)$ we obtain

$$a_\mu(-k) = -i(\sigma \cdot \mathbf{n})_{\mu\lambda}b_\lambda^\dagger(k). \quad (142)$$

¹²In the certain basis it is considered that the charge conjugation operator is just the complex conjugation operator for 4-vectors A_μ .

Thus, the above equations are self-consistent.

In the $(1, 0) \oplus (0, 1)$ representation we have somewhat different situation. Namely,

$$a_\mu(k) = [1 - 2(\mathbf{S} \cdot \mathbf{n})^2]_{\mu\lambda} a_\lambda(-k). \quad (143)$$

This signifies that in order to construct the Sankaranarayanan-Good field operator (which was used by Ahluwalia, Johnson and Goldman [Ahluwalia2], it satisfies $[\gamma_{\mu\nu} \partial_\mu \partial_\nu - \frac{(i\partial/\partial t)}{E} m^2] \Psi = 0$, we need additional postulates.

We can set for the 4-vector field operator:

$$\sum_\lambda \epsilon_\mu(-k, \lambda) a_\lambda(-k) = \sum_\lambda \epsilon_\mu^*(k, \lambda) b_\lambda^\dagger(k), \quad (144)$$

multiply both parts by $\epsilon_\nu[\gamma_{44}]_{\nu\mu}$, and use the normalization conditions for polarization vectors.

However, in the $(\frac{1}{2}, \frac{1}{2})$ representation we can also expand (apart the equation (144)) in the different way:

$$\sum_\lambda \epsilon_\mu(-k, \lambda) a_\lambda(-k) = \sum_\lambda \epsilon_\mu(k, \lambda) a_\lambda(k). \quad (145)$$

From the first definition we obtain (the signs \mp depends on the value of σ):

$$b_\sigma^\dagger(k) = \mp \sum_{\mu\nu\lambda} \epsilon_\nu(k, \sigma) [\gamma_{44}]_{\nu\mu} \epsilon_\mu(-k, \lambda) a_\lambda(-k), \quad (146)$$

or

$$b_\sigma^\dagger(k) = \frac{E_k^2}{m^2} \begin{pmatrix} 1 + \frac{\mathbf{k}^2}{E_k^2} & \sqrt{2} \frac{k_r}{E_k} & -\sqrt{2} \frac{k_l}{E_k} & -\frac{2k_3}{E_k} \\ -\sqrt{2} \frac{k_r}{E_k} & -\frac{k_r^2}{\mathbf{k}^2} & -\frac{m^2 k_3^2}{E_k^2 \mathbf{k}^2} + \frac{k_r k_l}{E_k^2} & \frac{\sqrt{2} k_3 k_r}{\mathbf{k}^2} \\ \sqrt{2} \frac{k_l}{E_k} & -\frac{m^2 k_3^2}{E_k^2 \mathbf{k}^2} + \frac{k_r k_l}{E_k^2} & -\frac{k_l^2}{\mathbf{k}^2} & -\frac{\sqrt{2} k_3 k_l}{\mathbf{k}^2} \\ \frac{2k_3}{E_k} & \frac{\sqrt{2} k_3 k_r}{\mathbf{k}^2} & -\frac{\sqrt{2} k_3 k_l}{\mathbf{k}^2} & \frac{m^2}{E_k^2} - \frac{2k_3}{\mathbf{k}^2} \end{pmatrix} \begin{pmatrix} a_{00}(-k) \\ a_{11}(-k) \\ a_{1-1}(-k) \\ a_{10}(-k) \end{pmatrix}. \quad (147)$$

From the second definition $\Lambda_{\sigma\lambda}^2 = \mp \sum_{\nu\mu} \epsilon_\nu^*(k, \sigma) [\gamma_{44}]_{\nu\mu} \epsilon_\mu(-k, \lambda)$ we have:

$$a_\sigma(k) = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & \frac{k_3^2}{k^2} & \frac{k_l^2}{k^2} & \frac{\sqrt{2}k_3k_l}{k^2} \\ 0 & \frac{k_r^2}{k^2} & \frac{k_3^2}{k^2} & -\frac{\sqrt{2}k_3k_r}{k^2} \\ 0 & \frac{\sqrt{2}k_3k_r}{k^2} & -\frac{\sqrt{2}k_3k_l}{k^2} & 1 - \frac{2k_3^2}{k^2} \end{pmatrix} \begin{pmatrix} a_{00}(-k) \\ a_{11}(-k) \\ a_{1-1}(-k) \\ a_{10}(-k) \end{pmatrix}. \quad (148)$$

It is the strange case: the field operator will only destroy particles (like in the $(1, 0) \oplus (0, 1)$ case). Possibly, we should think about modifications of the Fock space in this case, or introduce several field operators for the $(\frac{1}{2}, \frac{1}{2})$ representation.

However, other way is possible: to construct the left- and right- parts of the $(1, 0) \oplus (0, 1)$ field operator separately each other. In this case the commutation relations may be more complicated.

Finally, going back to the rest $(S, 0) \oplus (0, S)$ objects. **Bogoliubov constructs them introducing the products with delta functions like $\delta(k_0 - m)$. Then, he makes the boost of the "spinors" only, and changes by hand the δ to $\delta(k^2 - m^2)$ (where we already have $k_0 = E = \sqrt{k^2 + m^2}$). Mathematicians did not answer me, how can it be possible to make the boost of the delta functions consistently in such a way.**

The conclusion is: we still have few questions unsolved in the bases of the quantum field theory, which open a room for generalized theories.

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